QUESTION

Write down the matrix A corresponding to the quadratic form $5x^2 + 5y^2 + 2z^2 - 2xy$. Find its eigenvalues and the corresponding normalised eigenvectors, and write down a matrix R such that R^TAR is diagonal with the eigenvalues of A as its diagonal entries. Show that this is so by calculating R^TA and $(R^TA)R$. Hence or otherwise diagonalise the quadratic form, i.e. write it in the form $aX^2+bY^2+cZ^2$ giving explicit values for a, b, c and give expressions for the new variables X, Y, Z in terms of the variables x, y, z.

ANSWER

$A = \begin{pmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\det(A-\lambda I) = (2-\lambda)\left[(5-\lambda)^2 - 1\right]$ so $\lambda = 2$ or $(5-\lambda-1)(5-\lambda+1) = 0 \Rightarrow \lambda = 2$ or $\lambda = 4$ or $\lambda = 6$.

Eigenvectors:

$\lambda = 2$

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

so $\left.\begin{matrix} 3x = y \\ x = 3y \end{matrix}\right\} \Rightarrow x = y = 0 \; z = 1 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 4$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

$\Rightarrow x = y \; z = 0 \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$

$\lambda = 6$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

$\Rightarrow x = y = 0 \; z = 1 \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$

Let $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so $R^T = R$

$$R^T A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} & 0 \\ 3\sqrt{2} & -3\sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$(R^T A)R = \begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} & 0 \\ 3\sqrt{2} & -3\sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The quadratic form diagonalises to $4X^2 + 6Y^2 + 2Z^2$ where $X = \frac{1}{\sqrt{2}}(x + y),\ Y = \frac{1}{\sqrt{2}}(x - y),\ z = Z$.