QUESTION

(b) The life (in months) of an industrial product is a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 192(x+4)^{-4}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that the distribution function is $F_X(x) = 1 - \dfrac{64}{(x+4)^3}$ for $x > 0$.

(ii) Find the probability that the life of the product is more than 4 months.

(iii) Find the mean life of the product.

ANSWER

(b)

$$f_X(x) = \begin{cases} \frac{192}{(x+4)^4} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Distribution function

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(x)\,dx \\ &= \int_{-\infty}^{0} 0\,dx + \int_0^x \frac{192}{(x+4)^4}\,dx \\ &= 192\left[\frac{(x+4)^{-3}}{-3}\right]_0^x \\ &= 64\left\{-\frac{1}{(x+4)^3} - (-\frac{1}{4^3})\right\} \\ F_X(x) &= 1 - \frac{64}{(x+4)^3},\ x > 0 \end{aligned}$$

(ii) $P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - \left(1 - \dfrac{64}{(8)^3}\right) = \dfrac{64}{8^3} = \dfrac{1}{8}$

(iii)

$$\begin{aligned} \text{Mean} &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_0^{\infty} x \frac{192}{(x+4)^4}\, dx \end{aligned}$$

$$\int_0^X \frac{192x}{(x+4)^4} dx = 192 \int_0^X \frac{x+4-4}{(x+4)^4}\, dx$$

$$\begin{aligned} &= 192 \int_0^X \left(\frac{1}{(x+4)^3} - \frac{4}{(x+4)^4} \right) dx \\ &= 192 \left[\frac{(x+4)^{-2}}{-2} - \frac{4(x+4)^{-3}}{-3} \right]_0^X \\ &= 192 \left(\frac{-1}{2(X+4)^2} + \frac{4}{3(X+4)^3} + \frac{1}{2(4^2)} - \frac{4}{3(4^3)} \right) \end{aligned}$$

Hence as $X \to \infty$ mean$= 0 + 192\left(\frac{1}{32} - \frac{1}{48}\right) = 2$.
So the mean is 2 weeks.