Question
Use integration by parts to show that if

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \; n \geq 2,$$

then

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

(Hint: Write $\tan^n x = \tan^{n-2} x \tan^2 x$ and use an appropriate trigonometric identity for $\tan^2 x$).
Evaluate I_0. Hence show that

$$I_8 = \frac{\pi}{4} - \frac{76}{105}$$

Answer

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{4}} \tan^n x \, dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx \\
& \cos^2 x + \sin^2 x = 1 \Rightarrow 1 + \tan^2 x = \sec^2 x \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx \\
&= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - I_{n-2}
\end{aligned}$$

set $J = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx$

$$v = \tan^{n-2} x; \quad \frac{du}{dx} = \sec^2 x$$

$$\frac{dv}{dx} = (n-2) \tan^{n-2} x \sec^2 x; \quad u = \tan x$$

Integrate by parts:

$$\begin{aligned}
J &= [\tan^{n-2}x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (n-2)\tan^{n-2}x\sec^2 x\tan x \\
&= 1-(n-2)\int_0^{\frac{\pi}{4}} \tan^{n-2}x\sec^2 x\,dx \\
\Rightarrow J &= 1-(n-2)J \\
\Rightarrow J &= \underline{\frac{1}{n-1}}
\end{aligned}$$

Thus

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$\Rightarrow \underline{I_n + I_{n-2} = \frac{1}{n-1}}$$

as required.

$$I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x\,dx = \frac{\pi}{4}$$

so

$$\begin{aligned}
I_{2n} &= \frac{1}{2n-1} - I_{2n-2} \\
I_{2n-1} &= \frac{1}{2n-2} - I_{2n-4} \\
I_{2n-4} &= \frac{1}{2n-5} - I_{2n-6} \\
&\vdots \\
I_8 &= \frac{1}{-}7 - I_6 \\
I_6 &= \frac{1}{5}I_4 \\
I_4 &= \frac{1}{3} - I_2 \\
I_2 &= 1 - I_0 \\
&= 1 - \frac{\pi}{4}
\end{aligned}$$

back substitute:

$$\begin{aligned}
I_4 &= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \frac{1}{3} - 1 + \frac{\pi}{4} \\
I_6 &= \frac{1}{5} - \left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} \\
I_8 &= \frac{1}{7} - \left(\frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}\right) = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{\pi}{4} - \frac{76}{10}
\end{aligned}$$

as required.