Partial Differentiation
Limits

Question
Explain how the function

$$f(x,y) = \frac{x^2 + y^2 - x^3y^3}{x^2 + y^2}, \quad (x,y) \neq (0,0)$$

can be defined at $(0,0)$, so that it becomes continuous at all points of the xy-plane.

Answer

$$f(x,y) = \frac{x^2 + y^2 - x^3y^3}{x^2 + y^2} = 1 - \frac{x^3y^3}{x^2 + y^2}$$

But

$$\left|\frac{x^3y^3}{x^2 + y^2}\right| = \left|\frac{x^2}{x^2 + y^2}\right| \left|xy^3\right| \leq \left|xy^3\right| \to 0$$

as $(x,y) \to (0,0)$.

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 1 - 0 = 1.$$

So define $f(0,0) = 1$.