Question

Let X be a discrete random variable with probability generating function $G(s)$. Show that

(i) $G(1) = 1$

(ii) $G'(1) = E(X)$

(iii) $G''(1) = E(X(X-1))$

(iv) $\mathrm{Var}(X) = G''(1) + G'(1) - (G(1))^2$

Answer

Let (p_n) be the probability distribution of X

$$\begin{array}{rclrcl} G(s) & = & \sum_{n=0}^{\infty} p_n s^n & G(1) & = & \sum p_n = 1 \\ G'(s) & = & \sum_{n=1}^{\infty} n p_n s^{n-1} & G'(1) & = & \sum_{n=1}^{\infty} n p_n = E(X) \\ G''(s) & = & \sum_{n=2}^{\infty} n(n-1) p_n s^{n-2} & G''(1) & = & \sum_{n=2}^{\infty} n(n-1) p_n \\ & & & & = & E(X(X-1)) \end{array}$$

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= E(X(X-1)) + E(X) - E(X)^2 \\ &= G''(1) + G'(1) - (G'(1))^2 \end{aligned}$$