Question

Show that the roots of $x^4+(4-\varepsilon)x^3+(6-2\varepsilon)x^2+(4+\varepsilon)x^2+(4+\varepsilon)x+1-\varepsilon^2=0$ have the form

$$x=-1+(2\varepsilon)^{\frac{1}{4}}e^{i\pi\frac{(2n+1)}{4}}+O\left(\varepsilon^{\frac{1}{2}}\right),\ n=1,2,3,4,\ \varepsilon\to 0.$$

Answer

$x^4+(4-\varepsilon)x^3+*(6+2\varepsilon)x^2+(4+\varepsilon)x+1-\varepsilon^2=0$
Put $\varepsilon=0$: $x^4+4x^3+6x^2+4x+1=0$
Pascal's Triangle should come to mind (look at 1,4,6,4,1 pattern).
Thus we have $(x+1)^4=0$ i.e., $x=-1$ four times degeneracy.
So try ansatz

$$x=x_0+x_1\varepsilon^{\frac{1}{4}}+x_2\varepsilon^{\frac{1}{2}}+O(\varepsilon^{\frac{3}{4}})$$

and substitute.
$[x_0+x_1\varepsilon^{\frac{1}{4}}+x_2\varepsilon^{\frac{1}{2}}+O(\varepsilon^{\frac{3}{4}})]^4+(4-\varepsilon)(x_0+\varepsilon^{\frac{1}{4}}x_1+\varepsilon^{\frac{1}{2}}x_2+O(\varepsilon^{\frac{3}{4}})^3$
$+(6+2\varepsilon)(x_0+\varepsilon^{\frac{1}{4}}x_1+\varepsilon^{\frac{1}{2}}x_2+O(\varepsilon^{\frac{3}{4}}))^2+(4+\varepsilon)(x_0+\varepsilon^{\frac{1}{4}}+\varepsilon^{\frac{1}{2}}x_2+O(\varepsilon^{\frac{3}{4}})+1-\varepsilon^2=0$
$x_0^4+4x_0^3x_1\varepsilon^{\frac{1}{4}}+6x_0^2x_1^2\varepsilon^{\frac{1}{2}}+4x_0^3x_2\varepsilon^{\frac{1}{2}}+O(\varepsilon^{\frac{3}{4}})$
$+(4-\varepsilon)(x_0^3+3x_0^2\varepsilon^{\frac{1}{4}}x_1+3x_0\varepsilon^{\frac{1}{2}}x_1^2+3x_0^2\varepsilon^{\frac{1}{2}}x_2+O(\varepsilon^{\frac{3}{4}})$
$+(6+2\varepsilon)(x_0^2+2x_0x_1\varepsilon^{\frac{1}{4}}+2x_0x_2\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{1}{2}}x_2^2+O(\varepsilon^{\frac{3}{4}}))$
$+(4+\varepsilon)(x_0+\varepsilon^{\frac{1}{4}}x_1+\varepsilon^{\frac{1}{2}}x_2+O(\varepsilon^{\frac{3}{4}}))$
$+1-\varepsilon^2$
Balance at
$O(\varepsilon^0): x_0^4+4x_0^3+6x_0^2+4x_0+1=0 \Rightarrow$ roots $-1(\times 4)$ as above
$O(\varepsilon^{\frac{1}{4}}): 4x_0^3x_1+12x_0^2x_1+12x_0x_1+4x_1 \Rightarrow$ put $\alpha x_0=-1$ and see
$0=-4x_1+12x_1-12x_1+4x_1$. Therefore no information on x_1. Must go to $O(\varepsilon^{\frac{1}{2}})$.
$O(\varepsilon^{\frac{1}{2}}): 6x_0^2x_1^2+4x_0^3x_2+12x_0x_1^2+12x_0^2x_2+12x_0x_2+4x_2+6x_1^2=0$
$x_0=-1 \Rightarrow 6x_1^2-4x_2-12x_1^2+12x_2-12x_2+4x_2+6x_1^2=0$
$\underline{0=0}$!!
What's happening? It's just an extreme version of what we have met before: degenerate roots lead to extra work in the sense that you have to go to higher orders. In fact $O(\varepsilon^{\frac{3}{4}})$ also leads to 0=0. Only when you get to $O(\varepsilon)$ can you get: $2+x_1^4=0 \Rightarrow x_1=2^{\frac{1}{4}}e^{-\frac{\pi}{4}(2n+1)}$.
Thus $x=-1+2^{\frac{1}{4}}e^{\frac{i\pi(2n+1)}{4}}\varepsilon^{\frac{1}{4}}+O(\varepsilon^{\frac{1}{2}}),\ n=0,1,2,3$
To check this just substitute back into equation and observe the cancellations up to $O(\varepsilon^{\frac{1}{4}})$.