Exam Question

Topic: Laplace

In this question $L(y(x))$ denotes the Laplace transform of $y(x)$.

(i) Show that $L\left(e^{-ax}\cos bx\right) = \dfrac{p+a}{(p+a)^2+b^2}$ for $p > -a$.

(ii) Show that $L\left(e^{-ax}\sin bx\right) = \dfrac{b}{(p+a)^2+b^2}$ for $p > -a$.

(iii) Find $y(x)$ if $L(y(x)) = \dfrac{p}{p^2-4p+13}$.

Solution

$$\text{Let}\quad I_1 = L\left(e^{-ax}\cos bx\right) = \int_0^\infty e^{-(p+a)x}\cos bx\,dx$$

$$\text{Let}\quad I_2 = L\left(e^{-ax}\sin bx\right) = \int_0^\infty e^{-(p+a)x}\sin bx\,dx$$

$$\begin{aligned}
bI_1 &= \left[e^{-ax}\sin bx\right]_0^\infty + (p+a)I_2 = (p+a)I_2 \\
&\Rightarrow\ bI_1 - (p+a)I_2 = 0 && (1)\\
-bI_2 &= \left[e^{-ax}\cos bx\right]_0^\infty + (p+a)I_1 = -1 + (p+a)I_1 \\
&\Rightarrow\ (p+a)I_1 + bI_2 = 1 && (2)
\end{aligned}$$

$$b\times(1) + (p+a)\times(2)\quad\text{gives}\quad \left[(p+a)^2+b^2\right]I_1 = (p+a)$$

$$\Rightarrow I_1 = \frac{p+a}{(p+a)^2+b^2}\quad\text{and so}\quad I_2 = \frac{b}{(p+a)^2+b^2}$$

$$\begin{aligned}
L(y(x)) &= \frac{p}{p^2-4p+13} = \frac{p-2}{(p-2)^2+3^2} + \frac{2}{(p-2)^2+3^2},\\
\text{so}\quad y(x) &= e^{2x}\left(\cos 3x + \tfrac{2}{3}\sin 3x\right).
\end{aligned}$$