Question

Find the orthogonal trajectories to the following one-parameter families of curves (c is the parameter that is constant on each curve). In each case sketch some of the curves and their orthogonal trajectories.

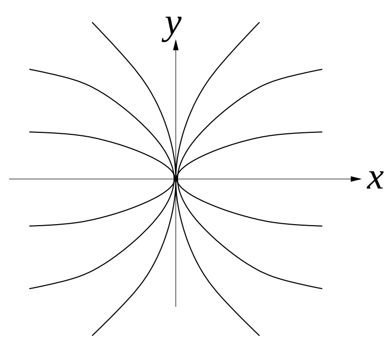
1.
$$y^2 = 4cx$$

2.
$$y = ce^{-x} \qquad (*)$$

3.
$$x = ce^{-y^2} \qquad (*)$$

Answer

a) $y^2 = 4cx \Rightarrow$ these are parabolas.



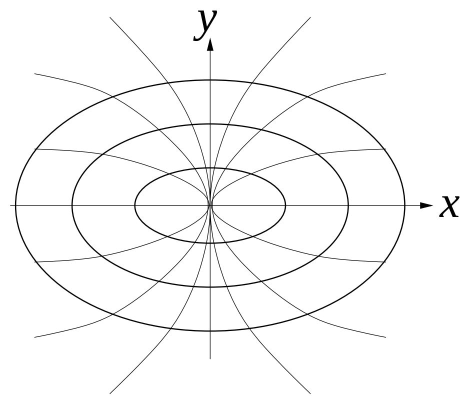
ODE of curves $\begin{cases} c = \frac{y^2}{4x} \\ 2y\frac{dy}{dx} = 4c \end{cases}$

$\Rightarrow 2y\dfrac{dy}{dx} = \dfrac{4y^2}{4x} \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{2x}$

ODE for orthogonal trajectory is $-\dfrac{1}{\left(\frac{dy}{dx}\right)} = \dfrac{y}{2x} \Rightarrow \dfrac{dy}{dx} = -\dfrac{2x}{y}$

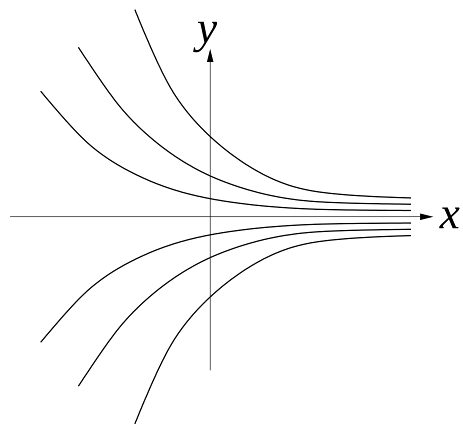
$\Rightarrow \displaystyle\int y dy = -\int 2x dx \Rightarrow \frac{1}{2}y^2 = -x^2 + A \Rightarrow x^2 + \frac{1}{2}y^2 = A$

these are ellipses centred at (0,0).



(ellipses represent orthogonal trajectories)

b) $y = ce^{-x}$

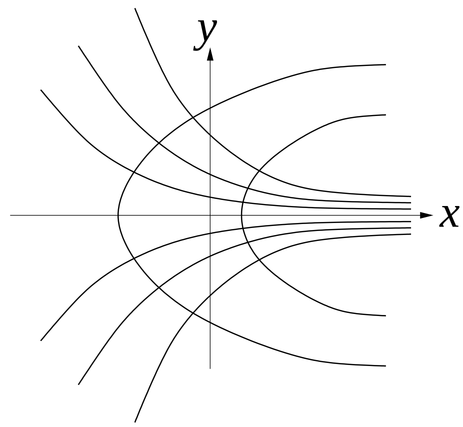


ODE of curves $\begin{cases} c = e^x y \\ \frac{dy}{dx} = -ce^{-x} \end{cases}$

$$\Rightarrow \frac{dy}{dx} = -e^x y e^{-x} \Rightarrow \frac{dy}{dx} = -y$$

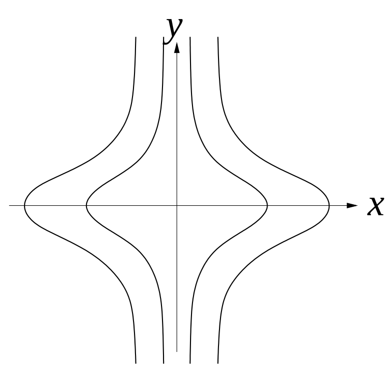
Orthogonal trajectories satisfy $-\frac{1}{\left(\frac{dy}{dx}\right)} = -y \Rightarrow \frac{dy}{dx} = \frac{1}{y}$

$\Rightarrow \int y dy = \int dx \Rightarrow \frac{1}{2}y^2 = x + d$ these are parabolas.



(parabolas represent orthogonal trajectories)

c) $x = ce^{-y^2}$

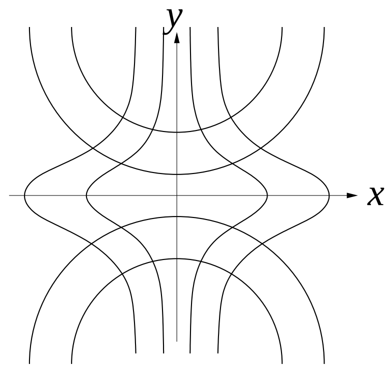


ODE of curves $\begin{cases} c = xe^{-y^2} \\ 1 = -2cy\frac{dy}{dx}e^{-y^2} \end{cases}$

$\Rightarrow 1 = -2xe^{y^2}y\dfrac{dy}{dx}e^{-y^2} \Rightarrow 1 = -2xy\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{2xy}$

ODE for orthogonal trajectory $\dfrac{-1}{\frac{dy}{dx}} = \dfrac{-1}{2xy} \Rightarrow \dfrac{dy}{dx} = 2xy$

$\int \frac{1}{y}dy = \int 2xdx \Rightarrow \ln y = x^2 + A \Rightarrow y = Be^{x^2}$



(top and bottom curves represent orthogonal trajectories)