Question

Find the general solution of the following differential equations

(i) $\dfrac{dy}{dx} + y = x$

(ii) $x\dfrac{dy}{dx} = 2x - y$

(iii) $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2 - 3$

(iv) $\sin x\left(\dfrac{dy}{dx}\sin x + 2y\cos x\right) = \sec^2 x$

(v) $x^2\dfrac{dy}{dx} + 2xy = 3\sin x + 1$

(vi) $\tan x\dfrac{dy}{dx}\dfrac{dy}{dx} + y = -y\tan^2 x$

Answer

(i) $\dfrac{dy}{dx} + y = x$

Not variables separable. Not homogeneous. Possibly exact.

LHS=$\dfrac{d}{dx}(xy)$

Thus

$$\begin{aligned} \frac{d}{dx}(xy) &= x \\ \Rightarrow \qquad xy &= \int x\,dx + c \end{aligned}$$

$\Rightarrow \underline{xy = \dfrac{x^2}{2} + c}$

(ii) $x\dfrac{dy}{dx} = 2x - y$

Solve as a linear equation.

$x\dfrac{dy}{dx} + y = 2x \Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = 2(\star)$

of $\dfrac{dy}{dx} + Py = Q \Rightarrow P = \dfrac{1}{x},\ Q = 2$

Thus integrating factor is

$$\begin{aligned} R &= \exp\left(\int P\,dx\right) \\ R &= \exp\left(\int \frac{dx}{d}\right) \\ &= \exp(\ln x) \\ &= x \end{aligned}$$

Thus multiplying $(\star)$ by R we have:

$$\begin{aligned} x\frac{dy}{dx} + x\frac{y}{x} &= 2x \\ \Rightarrow \quad x\frac{dy}{dx} + y &= 2x \end{aligned}$$

(what we started off with!)

Thus from notes

$$\frac{d}{dx}(xy) = 2x \Rightarrow \underline{xy = x^2 + c}$$

(iii) $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2 - 3 \ \ (\star)$ of $\dfrac{dy}{dx} + Py = Q$

$\Rightarrow P = -\dfrac{1}{x},\ Q = x^2 - 3$

Thus integrating factor is $R = \exp\left(\int -\dfrac{1}{x}\,dx\right) = \exp(-\ln x) = \dfrac{1}{x}$

Thus $\times(\star)$ by R

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = (x^2 - 3)\frac{1}{x}$$

$$\begin{aligned}
\text{or} \quad \frac{d}{dx}\left(y\frac{1}{x}\right) &= \frac{(x^2-3)}{x} \\
\Rightarrow \quad \frac{y}{x} &= \int \frac{(x^2-3)}{x}\,dx + c \\
\Rightarrow \quad \frac{y}{x} &= \int x\,dx - 3\int \frac{dx}{x} + c \\
\Rightarrow \quad \frac{y}{x} &= \frac{x^2}{2} - 3\ln x + c
\end{aligned}$$

$\Rightarrow \underline{2y = x^3 - 6x\ln x + cx}$

(iv) $\sin x\left(\dfrac{dy}{dx}\sin x + 2y\cos x\right) = \sec^2 x$

$$\frac{dy}{dx} + 2y\frac{\cos x}{\sin x} = \frac{1}{\cos^2 x\sin^2 x} \quad (\star)$$

of $\dfrac{dy}{dx} + Py = Q \Rightarrow P = 2\dfrac{\cos x}{\sin x},\ Q = \dfrac{1}{\cos^2 x\sin^2 x}$

Thus integrating factor

$$R = \exp\left(\int 2\frac{\cos x}{\sin x}\,dx\right) = \exp(2\ln\sin x)$$

$2\displaystyle\int \cot x\,dx = 2\ln\sin x$ standard integral.

Thus

$$\begin{aligned}
R &= \exp(2\ln\sin x) \\
&= \exp(\ln[\sin x]^2) \\
&= \sin^2 x
\end{aligned}$$

Thus $\times(\star)$ by $\sin^2 x$ to get

$$\sin^2 x\frac{dy}{dx} + 2y\cos x\sin x = \frac{1}{\cos^2 x}$$

i.e., what we started off with! Thus from notes LHS is

$$\begin{aligned}
\frac{d}{dx}([\sin^2 x]y) &= \frac{1}{\cos^2 x} \\
\Rightarrow \quad y\sin^2 x &= \int \sec^2 x\,dx
\end{aligned}$$

$\Rightarrow \underline{y\sin^2 x = \tan x + c}$ (both standard integrals)

(v) $x^2\dfrac{dy}{dx} + 2xy = 3\sin x + 1$

This is exact or linear with LHS:

$$\begin{aligned} \frac{d}{dx}(x^2y) &= 3\sin x + 1 \\ \Rightarrow \qquad x^2y &= \int (3\sin x + 1)\,dx \end{aligned}$$

$\underline{x^2y = 3\cos x + x + c}$

(vi) $\tan x \dfrac{dy}{dx}\dfrac{dy}{dx} + y = -y\tan^2 x$

$\Rightarrow \dfrac{dy}{dx} = y\dfrac{(1+\tan^2 x)}{\tan x} = 0 \ (\star)$

cf $\dfrac{dy}{dx} + Py = Q$

This is linear:

$\Rightarrow P = \dfrac{1+\tan^2 x}{\tan x},\ Q = 0$

Integrating factor

$$\begin{aligned} R &= \exp\left(\int \frac{1+\tan^2 x}{\tan x}\,dx\right) \\ &= \exp\left(\int \cot x\,dx + \int \tan x\,dx\right) \\ &= \exp\left(\int \frac{\cos x}{\sin x}\,dx + \int \frac{\sin x}{\cos x}\,dx\right) \\ &= \exp(\ln(\sin x) - \ln(\cos x)) \\ &= \exp\left(\ln\left(\frac{\sin x}{\cos x}\right)\right) \\ &= \exp(\ln(\tan x)) \\ &= \tan x \end{aligned}$$

Thus multiplying $(\star)$ through by R we have

$$\begin{aligned} \tan x\frac{dy}{dx} = (1+\tan^2 x)y &= 0 \\ \Rightarrow \qquad \tan x\frac{dy}{dx} + \sec^2 xy &= 0 \\ \Rightarrow \qquad \frac{d}{dx}(y\tan x) &= 0 \end{aligned}$$

$\Rightarrow \underline{y\tan x = c}$