Question

Solve the equations

(i) $\frac{d^2y}{dx^2} = x + \sin x$

(ii) $\frac{d^2y}{dx^2} = 3x^2$ given that when $x = 1,\ y = 0,\ \frac{dy}{dx} = 3$.

Answer

(i) $\frac{d^2y}{dx^2} = x + \sin x$ This is of the form $\frac{d^2y}{dx^2} = f(x)$

so can be integrated directly

$$\frac{d^2y}{dx^2} = x + \sin x$$

$$\Rightarrow \frac{dy}{dx} = \int (x + \sin x)\, dx = \frac{x^2}{2} - \cos x + C$$

$$\Rightarrow y = \int \left(\frac{x^2}{2} - \cos x + C\right) dx = \frac{x^3}{6} - \sin x + Cx + D$$

$C,\ D$ arbitrary

$$\Rightarrow \underline{y = \frac{x^3}{6} - \sin x + Cx + D}$$

(ii) $\frac{d^2y}{dx^2} = 3x^2$ is of the form $\frac{d^2y}{dx^2} = f(x)$, so can be integrated directly.

$$\frac{d^2y}{dx^2} = 3x^2$$

$$\Rightarrow \frac{Dy}{dx} = \int 3x^2\, dx = x^3 + C$$

Now use 1st boundary condition to find C: $\frac{dy}{dx} = 3$ when $x = 1$

Thus $3 = 1^3 + C \Rightarrow C = 2$

Thus $\frac{dy}{dx} = x^3 + 2$

Thus $y = \int (x^3 + 2)\, dx = \frac{x^4}{4} + 2x + D$

Now use 2nd boundary condition to find D: $y = 0$ when $x = 1$

Thus $0 = \frac{1^4}{4} + 2 \times 1 + D \Rightarrow -\frac{9}{4}$

Thus the particular solution is:

$$\underline{y = \frac{x^4}{4} + 2x - \frac{9}{4}}$$