Question
If a car driver has had k accidents in the time interval $(0,\ t]$, then in the time interval $(t, t+\delta t)$ she has a probability of

(i) $(\beta+\gamma k)\delta t + o(\delta t)$ of having one accident,

(ii) $1-(\beta+\gamma k)\delta t + o(\delta t)$ of having no accidents,

(iii) $o(\delta t)$ of having more than one accident.

Let $P_k(t)$ denote the probability that the driver has k accidents in the time interval $(0, t]$. Show that the generating function

$$G(z,t) = \sum_{k=0}^{\infty} p_k(t) z^k$$

satisfies the differential equation

$$\frac{\partial G}{\partial t} + \gamma z(1-z)\frac{\partial G}{\partial z} + \beta(1-z)G(z,t) = 0,$$

and explain why $G(z,0) \equiv 1$.

Verify that

$$G(z,t) = e^{-\beta t}\left[1 - \left(1 - e^{-\gamma t}\right) z\right]^{-\frac{\beta}{\gamma}}$$

satisfies these equations.

Answer
The unusual arguments concerning conditional probabilities leads to (omitting $\sigma(\delta t)$ terms)

$$p_0(t+\delta t) = p_0(t)(1-\beta\delta t)$$

we deduce

$$p_0'(t) = -\beta p_0(t)$$

For $k > 0$

$$p_k(t+\delta t) = p_k(t)(1-(\beta+\gamma k)\delta t) + p_{k-1}(t)(\beta+\gamma(k+1)\delta t)$$

We deduce that

$$p_k'(t) = -(\beta+\gamma k)p_k(t) + (\beta+\gamma(k-1))p_{k-1}(t)$$

Now

$$\begin{aligned}
\frac{\partial G}{\partial t} &= \sum_{k=0}^{\infty} p_k'(t) z^k \\
&= -\beta \sum_{k=0}^{\infty} p_k(t) z^k - \gamma z \sum_{k=0}^{\infty} p_k(t) k z^{k-1} \\
&\quad +\beta z \sum_{k=1}^{\infty} p_{k-1}(t) z^{k-1} + \gamma z^2 \sum_{k-1}^{\infty} p_{k-1}(t)(k-1) z^{k-2} \\
&= -\beta \sum_{k=0}^{\infty} p_k(t) z^k - \gamma z^2 \sum_{k=0}^{\infty} p_k(t) k z^{k-1} \\
&\quad +\beta z \sum_{k=0}^{\infty} p_k(t) z^k + \gamma z^2 \sum_{k=0}^{\infty} p_k(t) k z^{k-1} \\
&= -\beta G - \gamma z \frac{\partial G}{\partial z} + \beta z G + \gamma z^2 \frac{\partial G}{\partial z}
\end{aligned}$$

Hence

$$\frac{\partial G}{\partial t} + \gamma z(1-z)\frac{\partial G}{\partial z} + \beta(1-z) = 0$$

Now $p_0(0) = 1$ and $p_k(0) = 0$ for $k \geq 1$ Thus

$$G(z, 0) \equiv 1$$

Let $G(z,t) = e^{-\beta t}[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}}$ Then $G(z,0) \equiv 1$, and,

$$\begin{aligned}
\frac{\partial G}{\partial z} &= e^{-\beta t}[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}\frac{\beta}{\gamma}(1-e^{-\gamma t}) \\
\frac{\partial G}{\partial t} &= -\beta e^{-\beta t}[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}} \\
&\quad +e^{-\beta t}[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}\frac{\beta}{\gamma}\cdot \gamma e^{-\gamma t} z
\end{aligned}$$

Thus

$$\begin{aligned}
&e^{\beta t}(\beta G - \beta z G + \frac{\partial G}{\partial t} + \gamma z \frac{\partial G}{\partial z} - \gamma z^2 \frac{\partial G}{\partial z}) \\
&= \beta[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}} - \beta z[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}} \\
&\quad -\beta[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}} + [1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}\beta z e^{-\gamma t} \\
&\quad +\beta z[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}(1-e^{-\gamma t}) \\
&\quad -\beta z^2[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}(1-e^{-\gamma t}) \\
&= -\beta z[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}[1-(1-e^{-\gamma t})z] \\
&\quad +\beta z[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1} - \beta z^2[1-(1-e^{-\gamma t})z]^{-\frac{\beta}{\gamma}-1}(1-e^{-\gamma t}) \\
&= 0
\end{aligned}$$

Hence the given equation satisfies the differential equation.