Question

By transforming co-ordinates reduce the following conics to standard form. Sketch them with the original x-y axis

(i) $194x^2 + 120xy + 313y^2 + 776x + 240y + 607 = 0$

(ii) $9x^2 + 6xy + y^2 = 4$

(iii) $54x^2 - 144xy + 96y^2 + 15y + 20x = 0$

(iv) $x^2 + y^2 - 2x + 4y + 9 = 0$

Answer

(i)

$$194x^2 + 120xy + 313y^2 + 776x + 240y + 607 = 0$$

$B^2 - 4AC = 120^2 - 4 \cdot 194 \cdot 194 \cdot 313 < 0$ Elliptical type.

Let $x = \xi + h$ and $y = \eta + k$

The equation becomes

$194(\xi+h)^2+120(\xi+h)(\eta+k)+313(\eta+k)^2+776(\xi+h)+240(\eta+k)+607 = 0$

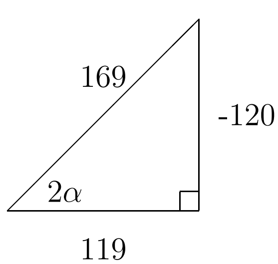
Choose h and k so that the linear terms in ξ and η vanish.

$$\left.\begin{array}{ll} \xi: & 338h + 120k + 776 = 0 \\ \eta: & 120h + 626k + 240 = 0 \end{array}\right. \Rightarrow k = 0,\ h = -2$$

The equation becomes

$$194\xi^2 + 120\xi\eta + 313\eta^2 - 169 = 0$$

Now rotate the axis through an angle alpha.



$$\begin{aligned} \xi &= X\cos\alpha - Y\sin\alpha \\ \eta &= X\sin\alpha + Y\cos\alpha \end{aligned}$$

Then the equation becomes:

$$\begin{aligned} X^2(194\cos^2\alpha + 120\cos\alpha\sin\alpha + 313\sin^2\alpha) \\ +Y^2(194\cos^2\alpha - 120\cos\alpha\sin\alpha + 313\sin^2\alpha) \\ +XY(120(\cos^2\alpha - \sin^2\alpha) + 238\sin\alpha\cos\alpha) - 169 &= 0 \end{aligned}$$

Choose α so that the coefficient of XY =0
i.e. $199\sin 2\alpha = -120\cos 2\alpha \Rightarrow \sin 2\alpha = -\dfrac{120}{169}\ \cos 2\alpha = \dfrac{119}{169}$
The the equation becomes

$$X^2 + 2Y^2 = 1$$

PICTURE

(ii)

$$9x^2 + 6xy + y^2 = 4$$

$B^2 - 4AC = 0$ Therefore we have a parabolic Type.
So rotate through angle α.

$$\begin{aligned} x &= X\cos\alpha - Y\sin\alpha \\ y &= X\sin\alpha + Y\cos\alpha \end{aligned}$$

The equation becomes

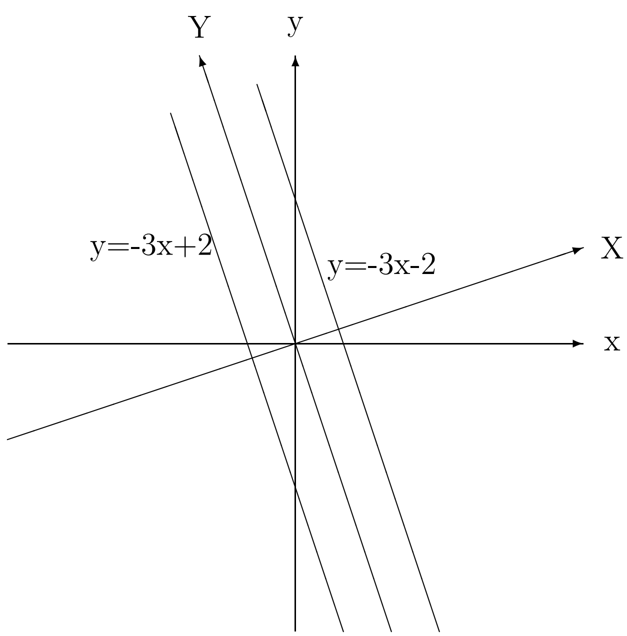
$$\begin{aligned} X^2(9\cos^2\alpha + 6\cos\alpha\sin\alpha + \sin^2\alpha) \\ +Y^2(9\sin^2\alpha - 6\sin\alpha\cos\alpha + \cos^2\alpha) \\ +XY(6\cos 2\alpha - 8\sin 2\alpha) &= 4 \end{aligned}$$

Choose α such that $\sin 2\alpha = \dfrac{6}{10}$, $\cos 2\alpha = \dfrac{8}{10}$

$$\text{So} \quad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha) = \frac{9}{10}$$
$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha) = \frac{1}{10}$$

Then the equation becomes

$$5X^2 = 2 \quad \text{Parallel lines.}$$



(iii)

$$54x^2 - 144xy + 96y^2 + 15y + 20x = 0$$

$B^2 - 4AC = 0$ Parabolic type.

$$x = X\cos\alpha - Y\sin\alpha$$
$$y = X\sin\alpha + Y\cos\alpha$$

We find that we need $7\sin 2\alpha = 24\cos 2\alpha$

$$sin2\alpha = \frac{24}{25}, \quad \cos 2\alpha = \frac{7}{25}$$

So $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha) = \frac{16}{25} \qquad \cos\alpha = \frac{4}{5},\ \sin\alpha = \frac{3}{5}$

The equation becomes

$$X + 6Y^2 = 0 \Rightarrow y^2 = -\frac{X}{6}$$

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(iv)

$$\begin{aligned} x^2 + y^2 - 2x + 4y + 9 &= 0 \\ (x-1)^2 + (y+2)^2 + 4 &= 0 \\ \text{No real locus} & \end{aligned}$$