Question

Find the tangential and normal acceleration of a particle which moves on the ellipse $\mathbf{r} = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}$.

Answer

$\dot{\mathbf{r}} = \omega(-a\sin\omega t\mathbf{i} + b\cos\omega t\mathbf{j})$

$\ddot{\mathbf{r}} = -\omega^2(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}) = -\omega^2\mathbf{r}$

The tangent unit vector is in the direction of $\mathbf{v}$

Therefore $\quad \mathbf{t} = \dfrac{(a\sin\omega t\mathbf{i} + b\cos\omega t\mathbf{j})}{\sqrt{a^2+b^2}}$

The normal unit vector is orthogonal to $\mathbf{t}$ and given from the Serret-Frenet formulae: $\dfrac{d\mathbf{t}}{ds} = +\kappa\mathbf{n}$

$\dfrac{d\mathbf{t}}{ds} = \dfrac{dt}{ds}\dfrac{d\mathbf{t}}{dt} = -\dfrac{dt}{ds}\dfrac{\omega}{\sqrt{a^2+b^2}}(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j})$

Therefore the unit vector parallel to $\dfrac{d\mathbf{t}}{ds} = \mathbf{u} = \dfrac{\mathbf{r}}{\sqrt{a^2+b^2}}$

The tangential component of acceleration is

$$\begin{aligned} & \mathbf{a}\cdot\mathbf{t} = -\omega^2\mathbf{r}\cdot\mathbf{t} \\ = \quad & -\omega^2(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j})\cdot\frac{-a\sin\omega t\mathbf{i} + b\cos\omega t\mathbf{j}}{\sqrt{a^2+b^2}} \\ = \quad & -\omega^2\frac{(b^2-a^2)}{\sqrt{a^2+b^2}} \end{aligned}$$

The normal component of acceleration is

$$\mathbf{a}\cdot\mathbf{n} = -\omega^2\mathbf{r}\cdot\frac{\mathbf{r}}{\sqrt{a^2+b^2}} = -\omega\frac{a^2\cos^2\omega t + b^2\sin^2\omega t}{\sqrt{a^2+b^2}}$$