QUESTION

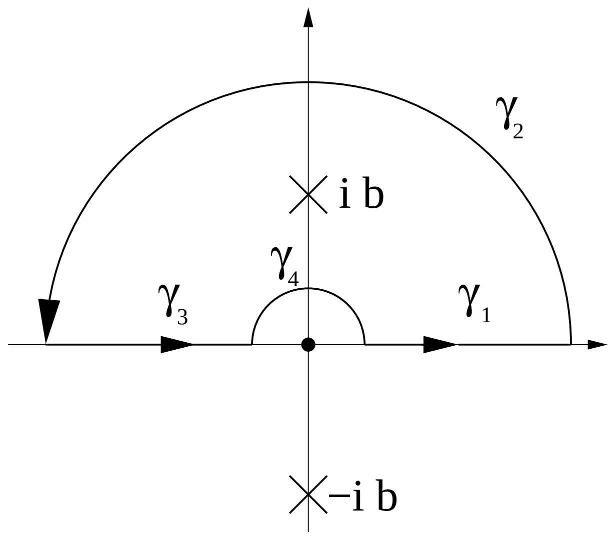
By integrating $\dfrac{z^a}{(z^2+b^2)}$ around a large semicircle indented at the origin, show that

$$\int_0^\infty \frac{x^a dx}{x^2+b^2} = \frac{\pi b^{a-1}}{2\cos\frac{\pi a}{2}},\ 0<a<1$$

ANSWER

$I = \displaystyle\int_0^\infty \frac{x^a}{x^2+b^2}\,dx \quad 0<a<1$

$J = \displaystyle\int_C \frac{z^a}{z^2+b^2}\,dz$



$$\begin{aligned}
C &= \gamma_1+\gamma_2+\gamma_3+\gamma_4 \\
\int_{\gamma_1} &= I \\
\left|\int_{\gamma_2}\right| &\le \frac{R^a}{r^2-b^2}\pi R \sim R^{a-1} \to 0 \\
&\quad \text{as } R\to\infty \text{ for } 0<a<1 \\
\int_{\gamma_3} &= \int_{-\infty}^0 \frac{z^a}{z^2+b^2}\,dz \\
&= -\int_0^\infty \frac{(-x)^a}{x^2+b^2}(-dx) \\
&\quad \text{(taking } z=-x \text{ and swapping the limits of integration)} \\
&= e^{i\pi a} I
\end{aligned}$$

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\begin{aligned}
\left|\int_{\gamma_4}\right| &\leq \frac{r^a}{b^2 - r^2}\pi r \sim r^{a+1} \to 0 \text{ as } r \to \infty \text{ for } 0 < a < 1 \\
J &= 2\pi i \operatorname{res}(ib) = 2\pi i \frac{(be^{i\frac{\pi}{2}})^a}{2ib} \\
&= \pi b^a e^{\frac{i\pi a}{2}} \\
I &= \frac{J}{1 + e^{i\pi a}} = \pi b^a \frac{e^{\frac{i\pi a}{2}}}{1 + e^{i\pi a}} \\
&= \pi b^a \frac{1}{e^{-\frac{i\pi a}{2}}} + e^{\frac{i\pi a}{2}} \\
&= \frac{\pi b^a}{2\cos\frac{\pi a}{2}}
\end{aligned}
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