Question

Viscous liquid of constant density ρ and constant kinematic viscosity ν is at rest in the region $o \leq y \leq h$ between two rigid parallel plates. There are no body forces. At time $t = 0$ the top plate is set into motion parallel to its own plane with speed U in the direction of the x-axis and is maintained at this speed thereafter. The plate at $y = 0$ is held fixed and the is no applied pressure gradient. Show that a flow solution of the form $\underline{q}(x,t) = (u(y,t), 0, 0)$ is possible provided u satisfies

$$u_t = \nu u_{yy}$$

and give suitable boundary and initial conditions for this equation.

Using seperation of variables, or otherwise, show that a solution to the governing partial differential equation is

$$u = C_1 y + C_2 + \sum_{n=1}^{\infty} e^{-k_n^2 t} \left(A_n \sin \frac{k_n}{\sqrt{\nu}} y + B_n \cos \frac{k_n}{\sqrt{\nu}} y \right)$$

where A_n, B_n, C_1, C_2 and k_n are constants. By futher imposing the boundary conditions, show that the solution for the flow is given by

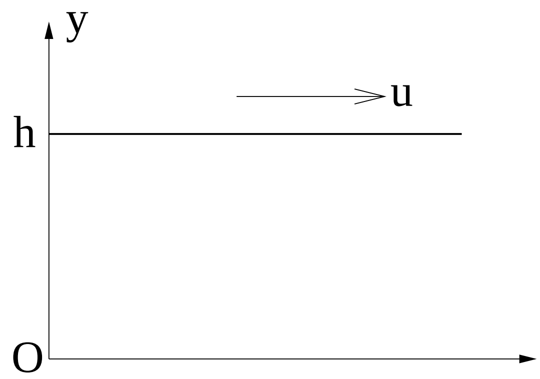
$$u = \frac{Uy}{h} + \frac{2U}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi y}{h}\right) \exp\left(-\frac{n^2\pi^2\nu t}{h^2}\right)$$

Explain briefly what you would expect the flow to look like for a very viscous fluid.

[You may use, without proof, the fact that the Fourier sine series representation of the function ξ for $\xi \in [0,1]$ is given by

$$\xi = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi\xi).]$$

Answer



Choose $\underline{q} = ((u(y,t)), 0, 0)$ then $div(\underline{q}) = 0$
Navier-Stokes equations become:

$$\begin{aligned} u_t + 0 &= -p_x/\rho + \nu(u_{xx} + u_{yy} + u_{zz}) \\ 0 &= p_y/\rho + 0 \\ 0 &= -p_z/\rho + 0 \end{aligned}$$

So since we are told that there are no pressure gradients

$$u_t = \nu u_{yy}$$

Initial conditions:-	$u(y,0) = 0$
Boundary condtions:-	$u(h) = 0,\ (t < 0), \quad u(h) = U\ (t \geq 0)$
Also by no slip	$u(0) = 0$

Now to use seperation of variables, set $u = Y(y)T(t)$.
Then

$$\begin{aligned} YT' &= \nu TY'' \\ \Rightarrow T'/T &= \nu Y''/Y \end{aligned}$$

By the standard separation of variables argument both sides must be either a constant or zero. Thus either
$\nu Y''/Y = 0 \quad \Rightarrow Y = C_1 y + C_2, \quad T = constant$
or
$T'/T = -k^2$ (choose constant -ve so solutions don't grow at $t = \infty$)

$$\Rightarrow T' + k^2 t = 0, \quad T = Ae^{-k^2 t}$$

Also $Y'' + \dfrac{k^2}{\nu} Y = 0 \quad \Rightarrow \quad Y = B\cos\dfrac{k}{\sqrt{\nu}}y + C\sin\dfrac{k}{\nu}y.$

Since the equation is linear, solutions may be added.

$$\Rightarrow u = C_1 y + C_2 + \sum_{n=1}^{\infty} e^{-k_n^2 t} \left(A_n \sin \frac{k_n}{\sqrt{\nu}} y + B_n \cos \frac{k_n}{\sqrt{\nu}} y \right)$$

(the term $n = 0$ just gives 0 and constant-see later).
Now we have to impose the boundary conditions:-
$u(0) = 0 \quad \Rightarrow$

$$C_2 = 0, B_n = 0 \quad \forall n$$

Thus

$$u = C_1 y + \sum_{n=1}^{\infty} A_n e^{-k_n^2 t} \sin \left(\frac{k_n}{\sqrt{\nu}} y \right).$$

Now the only way to have $u(y) = h \quad \forall t \geq 0$ is to have

$$\begin{aligned} c_1 &= \frac{U}{h} \quad \text{and} \\ \sin\left(\frac{k_n}{\sqrt{\nu}} h\right) &= 0, \quad \frac{k_n h}{\sqrt{\nu}} = n\pi \quad (n \in \mathbf{Z}) \end{aligned}$$

$$\begin{aligned} \Rightarrow k_n &= \sqrt{\nu} n\pi / h \quad (n \in \mathbf{Z}) \\ u &= \frac{Uy}{h} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{h}\right) \exp\left(-\frac{\nu n^2 \pi^2}{h^2} r\right) \end{aligned}$$

Finally we need $u = 0 \quad \forall y$ at $t = 0$.
$\Rightarrow 0 = \frac{Uy}{h} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right).$
From the result given in the question, for $\in [0, h]$ (set $x = fracyh$)

$$\frac{y}{h} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi y}{h}\right)$$

$$\begin{aligned} \Rightarrow A + n &= -\frac{u}{n\pi} 2(-1)^{n+1} \\ u &= \frac{Uy}{h} + \sum_{n=1}^{\infty} \frac{U}{n\pi} 2(-1)^n \sin\left(\frac{n\pi y}{h}\right) \exp\left(-\frac{n^2 \pi^2 \nu t}{h^2}\right) \end{aligned}$$

When ν is very large, the exponential terms would be very small for all but the smallest t. Thus for a very viscous fluid we would expect

$$u \sim \frac{Uy}{h} \quad \text{after a very short time.}$$